Algebraic Proof

SCAN ME

REVISE THIS TOPIC

1 Prove that $(n + 5)^2 - (3n + 8) = (n + 3)(n + 4) + 5$

$$(n+5)(n+5) - (3n+8)$$
$$= n^2 + 5n + 5n + 25 - 3n - 8$$
$$= n^2 + 7n + 17$$
$$= n^2 + 7n + 12 + 5$$
$$= (n+3)(n+4) + 5$$

(Total for Question 1 is 3 marks)

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2 Prove that $(2n - 1)^2 - (n - 3)^2 = (3n - 1)(n + 1) - 7$

$$(2n-1)(2n-1) - (n-3)(n-3)$$
$$= (4n^2 - 2n - 2n + 1) - (n^2 - 3n - 3n + 9)$$
$$= 4n^2 - 4n + 1 - n^2 + 3n + 3n - 9$$
$$= 3n^2 + 2n - 8$$
$$= 3n^2 + 2n - 1 - 7$$
$$= (3n-1)(n+1) - 7$$

(Total for Question 2 is 3 marks)

3 Prove that $(3n - 5)^2 - 2(4n - 5)(n - 3) = (n + 5)(n - 1)$

$(3n-5)(3n-5) - 2(4n-5)(n-3)$

$= (9n^2 - 15n - 15n + 25) - 2(4n^2 - 12n - 5n + 15)$

$= 9n^2 - 30n + 25 - 8n^2 + 24n + 10n - 30$

$= n^2 + 4n - 5$

$= (n+5)(n-1)$

(Total for Question 3 is 3 marks)

4 Prove that $(n - 3)^2 - (15 + n)(15 - n) = 2(n - 12)(n + 9)$

$(n-3)(n-3) - (15+n)(15-n)$

$= (n^2 - 3n - 3n + 9) - (225 - 15n + 15n - n^2)$

$= n^2 - 6n + 9 - 225 + 15n - 15n + n^2$

$= 2n^2 - 6n - 216$

$= 2(n^2 - 3n - 108)$

$= 2(n-12)(n+9)$

(Total for Question 4 is 3 marks)

5 n is an integer such that $n > 3$
Prove algebraically that $(n-2)^2 - (n-5)^2$ is always a multiple of 3.

$(n-2)(n-2) - (n-5)(n-5)$
$= (n^2 - 2n - 2n + 4) - (n^2 - 5n - 5n + 25)$
$= n^2 - 4n + 4 - n^2 + 5n + 5n - 25$
$= 6n - 21$
$= 3(2n - 7)$

$(2n-7)$ is an integer
so $3(2n-7)$ is a multiple of 3

(Total for Question 5 is 3 marks)

6 n is a positive integer.
Prove algebraically that $(3n+1)^2 - (3n-4)^2$ is always a multiple of 15.

$(3n+1)(3n+1) - (3n-4)(3n-4)$
$= (9n^2 + 3n + 3n + 1) - (9n^2 - 12n - 12n + 16)$
$= 9n^2 + 6n + 1 - 9n^2 + 12n + 12n - 16$
$= 30n - 15$
$= 15(2n - 1)$

$(2n-1)$ is an integer
so $15(2n-1)$ is a multiple of 15

(Total for Question 6 is 3 marks)

7 n is a positive integer.
Prove algebraically that $(2n + 5)^2 - (2n + 1)^2$ is always a multiple of 8

$(2n+5)(2n+5) - (2n+1)(2n+1)$
$= (4n^2 + 10n + 10n + 25) - (4n^2 + 2n + 2n + 1)$
$= 4n^2 + 20n + 25 - 4n^2 - 2n - 2n - 1$
$= 16n + 24$
$= 8(2n + 3)$

$(2n+3)$ is an integer
so $8(2n+3)$ is a multiple of 8

(Total for Question 7 is 3 marks)

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8 n is a positive integer such that $n > 2$
Prove algebraically that $(2n + 3)^2 + (3 - n)^2 - (n + 5)^2$ is always one more than a multiple of 4.

$(2n+3)(2n+3) + (3-n)(3-n) - (n+5)(n+5)$
$= (4n^2 + 6n + 6n + 9) + (9 - 3n - 3n + n^2) - (n^2 + 5n + 5n + 25)$
$= 4n^2 + 12n + 9 + 9 - 6n + n^2 - n^2 - 10n - 25$
$= 4n^2 - 4n - 7$
$= 4n^2 - 4n - 8 + 1$
$= 4(n^2 - n - 2) + 1$

$(n^2 - n - 2)$ is an integer so $4(n^2-n-2)$ is a multiple of 4. So $4(n^2-n-2)+1$ is one more than a multiple of 4.

(Total for Question 8 is 4 marks)

9 Prove algebraically that the sum of five consecutive positive integers is always a multiple of 5.

$(n) + (n+1) + (n+2) + (n+3) + (n+4)$

$= 5n + 10$

$= 5(n+2)$

$(n+2)$ is an integer

so $5(n+2)$ is a multiple of 5

(Total for Question 9 is 2 marks)

10 Arjan says: "The sum of four consecutive positive integers is always a multiple of 4".

Use an algebraic method to prove that Arjun is incorrect.

$(n) + (n+1) + (n+2) + (n+3)$

$= 4n + 6$

$4n+6$ is not a multiple of 4 since 6 is not a multiple of 4.

(Total for Question 10 is 2 marks)

11 Prove algebraically that the sum of six consecutive positive integers is always a multiple of 3.

$(n) + (n+1) + (n+2) + (n+3) + (n+4) + (n+5)$

$= 6n + 15$

$= 3(2n + 5)$

$2n + 5$ is an integer

so $3(2n+5)$ is a multiple of 3

(Total for Question 11 is 2 marks)

12 *n* is a positive integer.
Prove that $(4n - 3)^2 - 3(5n - 3)(n - 1)$ is always a square number.

$(4n-3)(4n-3) - 3(5n-3)(n-1)$

$= (16n^2 - 12n - 12n + 9) - 3(5n^2 - 3n - 5n + 3)$

$= 16n^2 - 24n + 9 - 15n^2 + 9n + 15n - 9$

$= n^2$

n^2 is a square number

(Total for Question 12 is 3 marks)

13 n is a positive integer.
Prove that $(3n + 1)(9n^2 – 3n + 1)$ is always 1 more than a cube number.

$$(3n+1)(9n^2-3n+1)$$
$$= 27n^3 - 9n^2 + 3n + 9n^2 - 3n + 1$$
$$= 27n^3 + 1$$
$$= (3n)^3 + 1$$

$(3n)^3$ is a cube number
so $(3n)^3 + 1$ is one more than a cube number

(Total for Question 13 is 4 marks)

14 n is a positive integer.
Prove that $(n + 2)^3 – n^3$ is always even.

$$(n+2)(n+2)(n+2) - n^3$$
$$= (n^2+4n+4)(n+2) - n^3$$
$$= n^3 + 4n^2 + 4n + 2n^2 + 8n + 8 - n^3$$
$$= n^3 + 6n^2 + 12n + 8 - n^3$$
$$= 6n^2 + 12n + 8$$
$$= 2(3n^2 + 6n + 4)$$

$(3n^2+6n+4)$ is an integer
so $2(3n^2+6n+4)$ is a multiple of 2
and is therefore even

(Total for Question 14 is 4 marks)

15 n is an integer.
Prove that $n^2 - 6n + 10$ is always positive.

$$n^2 - 6n + 10$$
$$= (n-3)^2 - 9 + 10$$
$$= (n-3)^2 + 1$$

$(n-3)^2 \geq 0$ and $1 > 0$

so $(n-3)^2 + 1 > 0$ (always positive)

(Total for Question 15 is 3 marks)

16 n is an integer.
Prove that $n^2 + 3n + 3$ is always positive.

$$n^2 + 3n + 3$$
$$= \left(n + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 3$$
$$= \left(n + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{12}{4}$$
$$= \left(n + \tfrac{3}{2}\right)^2 + \tfrac{3}{4}$$

$\left(n + \tfrac{3}{2}\right)^2 \geq 0$ and $\tfrac{3}{4} > 0$

so $\left(n + \tfrac{3}{2}\right)^2 + \tfrac{3}{4} > 0$ (always positive)

(Total for Question 16 is 3 marks)

17 n is an integer.
Prove that $2n - n^2 - 2$ is always negative.

$$2n - n^2 - 2 = -[n^2 - 2n + 2]$$
$$= -[(n-1)^2 - 1 + 2]$$
$$= -[(n-1)^2 + 1]$$
$$= -(n-1)^2 - 1$$

$-(n-1)^2 \leq 0$ and $-1 < 0$

so $-(n-1)^2 - 1 < 0$ (always negative)

(Total for Question 17 is 3 marks)

18 n and m are consecutive integers and $m > n$.
Prove algebraically that $m^2 - n^2$ is always an odd number.

Let $m = n+1$

$$m^2 - n^2 = (n+1)^2 - n^2$$
$$= (n+1)(n+1) - n^2$$
$$= n^2 + 2n + 1 - n^2$$
$$= 2n + 1$$

$2n$ is always even
so $2n+1$ is always odd.

(Total for Question 18 is 3 marks)

19 n and m are consecutive integers and $m > n$.
Prove algebraically that $mn + m$ is always a square number.

Let $m = n+1$

$$mn + m = (n+1)n + (n+1)$$
$$= n^2 + n + n + 1$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$

$(n+1)$ is an integer so $(n+1)^2$ is always a square number

(Total for Question 19 is 3 marks)

20 Prove algebraically that the sum of three consecutive even numbers is always a multiple of 6.

$(2n) + (2n+2) + (2n+4)$

$= 6n + 6$

$= 6(n+1)$

$(n+1)$ is an integer

so $6(n+1)$ is a multiple of 6

(Total for Question 20 is 2 marks)

21 Prove algebraically that the difference between the squares of two consecutive even numbers is always a multiple of 4

$(2n+2)^2 - (2n)^2$

$= 4n^2 + 4n + 4n + 4 - 4n^2$

$= 8n + 4$

$= 4(2n+1)$

$(2n+1)$ is an integer

so $4(2n+1)$ is a multiple of 4

(Total for Question 21 is 3 marks)

22 Prove algebraically that the sum of the squares of three consecutive integers is one less than a multiple of 3.

$(n)^2 + (n+1)^2 + (n+2)^2$

$= (n^2) + (n^2 + 2n + 1) + (n^2 + 4n + 4)$

$= 3n^2 + 6n + 5$

$= 3n^2 + 6n + 6 - 1$

$= 3(n^2 + 2n + 1) - 1$

$(n^2 + 2n + 1)$ is an integer

so $3(n^2+2n+1)$ is a multiple of 3

and $3(n^2 + 2n + 1) - 1$ is one less than a multiple of 3.

(Total for Question 22 is 4 marks)

23 Prove algebraically that the difference between the squares of consecutive integers is equal to the sum of the two integers.

Let $n, n+1$ be consecutive integers

$(n+1)^2 - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n + 1$

$= (n) + (n+1)$

$(n)+(n+1)$ is the sum of the two integers

(Total for Question 23 is 3 marks)

24 Prove algebraically that the product of two consecutive odd numbers is one less than a multiple of 4.

$(2n+1)(2n+3)$

$= 4n^2 + 2n + 6n + 3$

$= 4n^2 + 8n + 3$

$= 4n^2 + 8n + 4 - 1$

$= 4(n^2 + 2n + 1) - 1$

(n^2+2n+1) is an integer

so $4(n^2+2n+1)$ is a multiple of 4

and $4(n^2+2n+1) - 1$ is one less than a multiple of 4

(Total for Question 24 is 3 marks)

25 Prove algebraically that the product of three consecutive even numbers is always a multiple of 8.

$2n(2n+2)(2n+4)$

$= 2n(4n^2 + 8n + 4n + 8)$

$= 2n(4n^2 + 12n + 8)$

$= 8n^3 + 24n^2 + 16n$

$= 8(n^3 + 3n^2 + 2n)$

(n^3+3n^2+2n) is an integer

so $8(n^3+3n^2+2n)$ is a multiple of 8

(Total for Question 25 is 3 marks)

26 a and b are positive integers.
a is two more than a multiple of 5.
b is two less than a multiple of 5.

Prove that ab is one more than a multiple of 5.

Let $a = 5n + 2$ and $b = 5m - 2$

$ab = (5n+2)(5m-2)$

$= 25mn - 10n + 10m - 4$

$= 25mn - 10n + 10m - 5 + 1$

$= 5(5mn - 2n + 2m - 1) + 1$

$(5mn - 2n + 2m - 1)$ is an integer

so $5(5mn - 2n + 2m - 1)$ is a multiple of 5

and $5(5mn - 2n + 2m - 1) + 1$ is one more than a multiple of 5.

(Total for Question 26 is 4 marks)

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27 Prove that the sum of the squares of three consecutive integers is equal to five more than three times the product of the largest and smallest of the three integers.

Let $n, n+1, n+2$ be consecutive integers

$n^2 + (n+1)^2 + (n+2)^2$

$= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4)$

$= 3n^2 + 6n + 5$

$= 3(n)(n+2) + 5$

(n) is the smallest of the integers and $(n+2)$ is the largest so $3(n)(n+2) + 5$ is 5 more than 3 times their product.

(Total for Question 27 is 3 marks)