# Algebraic Proof

SCAN ME

REVISE THIS TOPIC

**1** Prove that $(n + 5)^2 - (3n + 8) = (n + 3)(n + 4) + 5$ **[3 marks]**

$(n+5)(n+5) - (3n+8)$

$= n^2 + 5n + 5n + 25 - 3n - 8$

$= n^2 + 7n + 17$

$= n^2 + 7n + 12 + 5$

$= (n+3)(n+4) + 5$

**2** Prove that $(2n - 1)^2 - (n - 3)^2 = (3n - 1)(n + 1) - 7$ **[3 marks]**

$(2n-1)(2n-1) - (n-3)(n-3)$

$= (4n^2 - 2n - 2n + 1) - (n^2 - 3n - 3n + 9)$

$= 4n^2 - 4n + 1 - n^2 + 3n + 3n - 9$

$= 3n^2 + 2n - 8$

$= 3n^2 + 2n - 1 - 7$

$= (3n-1)(n+1) - 7$

$\overline{\phantom{0}6}$

**3** Prove that $(3n-5)^2 - 2(4n-5)(n-3) = (n+5)(n-1)$ **[3 marks]**

$(3n-5)(3n-5) - 2(4n-5)(n-3)$

$= (9n^2 - 15n - 15n + 25) - 2(4n^2 - 12n - 5n + 15)$

$= 9n^2 - 30n + 25 - 8n^2 + 24n + 10n - 30$

$= n^2 + 4n - 5$

$= (n+5)(n-1)$

**4** Prove that $(n-3)^2 - (15+n)(15-n) = 2(n-12)(n+9)$ **[3 marks]**

$(n-3)(n-3) - (15+n)(15-n)$

$= (n^2 - 3n - 3n + 9) - (225 - 15n + 15n - n^2)$

$= n^2 - 6n + 9 - 225 + 15n - 15n + n^2$

$= 2n^2 - 6n - 216$

$= 2(n^2 - 3n - 108)$

$= 2(n-12)(n+9)$

**5** $n$ is an integer such that $n > 3$

Prove algebraically that $(n-2)^2 - (n-5)^2$ is always a multiple of 3. **[3 marks]**

$(n-2)(n-2) - (n-5)(n-5)$

$= (n^2 - 2n - 2n + 4) - (n^2 - 5n - 5n + 25)$

$= n^2 - 4n + 4 - n^2 + 5n + 5n - 25$

$= 6n - 21$

$= 3(2n - 7)$

$(2n-7)$ is an integer

so $3(2n-7)$ is a multiple of 3

**6** $n$ is a positive integer.

Prove algebraically that $(3n+1)^2 - (3n-4)^2$ is always a multiple of 15. **[3 marks]**

$(3n+1)(3n+1) - (3n-4)(3n-4)$

$= (9n^2 + 3n + 3n + 1) - (9n^2 - 12n - 12n + 16)$

$= 9n^2 + 6n + 1 - 9n^2 + 12n + 12n - 16$

$= 30n - 15$

$= 15(2n - 1)$

$(2n-1)$ is an integer

so $15(2n-1)$ is a multiple of 15

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**7** $n$ is a positive integer.
Prove algebraically that $(2n + 5)^2 - (2n + 1)^2$ is always a multiple of 8 **[3 marks]**

$(2n+5)(2n+5) - (2n+1)(2n+1)$

$= (4n^2 + 10n + 10n + 25) - (4n^2 + 2n + 2n + 1)$

$= 4n^2 + 20n + 25 - 4n^2 - 2n - 2n - 1$

$= 16n + 24$

$= 8(2n + 3)$

$(2n+3)$ is an integer

so $8(2n+3)$ is a multiple of 8

**8** $n$ is a positive integer such that $n > 2$
Prove algebraically that $(2n + 3)^2 + (3 - n)^2 - (n + 5)^2$ is always one more than a multiple of 4. **[4 marks]**

$(2n+3)(2n+3) + (3-n)(3-n) - (n+5)(n+5)$

$= (4n^2 + 6n + 6n + 9) + (9 - 3n - 3n + n^2) - (n^2 + 5n + 5n + 25)$

$= 4n^2 + 12n + 9 + 9 - 6n + n^2 - n^2 - 10n - 25$

$= 4n^2 - 4n - 7$

$= 4n^2 - 4n - 8 + 1$

$= 4(n^2 - n - 2) + 1$

$(n^2 - n - 2)$ is an integer so $4(n^2-n-2)$ is a multiple of 4. So $4(n^2-n-2)+1$ is one more than a multiple of 4.

**9** Prove algebraically that the sum of five consecutive positive integers is always a multiple of 5. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3) + (n+4)$

$= 5n + 10$

$= 5(n+2)$

$(n+2)$ is an integer

so $5(n+2)$ is a multiple of 5

**10** Arjan says:

*"The sum of four consecutive positive integers is always a multiple of 4".*

Use an algebraic method to prove that Arjun is incorrect. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3)$

$= 4n + 6$

$4n+6$ is not a multiple of 4 since 6 is not a multiple of 4.

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**11** Prove algebraically that the sum of six consecutive positive integers is always a multiple of 3. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3) + (n+4) + (n+5)$

$= 6n + 15$

$= 3(2n + 5)$

$2n+5$ is an integer

so $3(2n+5)$ is a multiple of 3

**12** $n$ is a positive integer.

Prove that $(4n - 3)^2 - 3(5n - 3)(n - 1)$ is always a square number. **[3 marks]**

$(4n-3)(4n-3) - 3(5n-3)(n-1)$

$= (16n^2 - 12n - 12n + 9) - 3(5n^2 - 3n - 5n + 3)$

$= 16n^2 - 24n + 9 - 15n^2 + 9n + 15n - 9$

$= n^2$

$n^2$ is a square number

**13** $n$ is a positive integer.
Prove that $(3n + 1)(9n^2 – 3n + 1)$ is always 1 more than a cube number. **[4 marks]**

$(3n+1)(9n^2-3n+1)$
$= 27n^3 - 9n^2 + 3n + 9n^2 - 3n + 1$
$= 27n^3 + 1$
$= (3n)^3 + 1$

$(3n)^3$ is a cube number
so $(3n)^3 + 1$ is one more than a cube number

**14** $n$ is a positive integer.
Prove that $(n + 2)^3 – n^3$ is always even. **[4 marks]**

$(n+2)(n+2)(n+2) - n^3$
$= (n^2+4n+4)(n+2) - n^3$
$= n^3 + 4n^2 + 4n + 2n^2 + 8n + 8 - n^3$
$= n^3 + 6n^2 + 12n + 8 - n^3$
$= 6n^2 + 12n + 8$
$= 2(3n^2 + 6n + 4)$
$(3n^2 + 6n + 4)$ is an integer
so $2(3n^2+6n+4)$ is a multiple of 2
and is therefore even

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**15** $n$ is an integer.
Prove that $n^2 - 6n + 10$ is always positive. **[3 marks]**

$n^2 - 6n + 10$
$= (n-3)^2 - 9 + 10$
$= (n-3)^2 + 1$

$(n-3)^2 \geqslant 0$ and $1 > 0$
so $(n-3)^2 + 1 > 0$ (always positive)

**16** $n$ is an integer.
Prove that $n^2 + 3n + 3$ is always positive. **[3 marks]**

$n^2 + 3n + 3$
$= (n + \frac{3}{2})^2 - \frac{9}{4} + 3$
$= (n + \frac{3}{2})^2 - \frac{9}{4} + \frac{12}{4}$
$= (n + \frac{3}{2})^2 + \frac{3}{4}$
$(n + \frac{3}{2})^2 \geqslant 0$ and $\frac{3}{4} > 0$
so $(n + \frac{3}{2})^2 + \frac{3}{4} > 0$ (always positive)

**17** $n$ is an integer.
Prove that $2n - n^2 - 2$ is always negative. **[3 marks]**

$2n - n^2 - 2 = -[n^2 - 2n + 2]$
$= -[(n-1)^2 - 1 + 2]$
$= -[(n-1)^2 + 1]$
$= -(n-1)^2 - 1$
$-(n-1)^2 \leqslant 0$ and $-1 < 0$
so $-(n-1)^2 - 1 < 0$ (always negative)

**18** $n$ and $m$ are consecutive integers and $m > n$.
Prove algebraically that $m^2 - n^2$ is always an odd number. **[3 marks]**

Let $m = n+1$

$m^2 - n^2 = (n+1)^2 - n^2$

$= (n+1)(n+1) - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n + 1$

$2n$ is always even
so $2n+1$ is always odd.

**19** $n$ and $m$ are consecutive integers and $m > n$.
Prove algebraically that $mn + m$ is always a square number. **[3 marks]**

Let $m = n+1$

$mn + m = (n+1)n + (n+1)$

$= n^2 + n + n + 1$

$= n^2 + 2n + 1$

$= (n+1)^2$

$(n+1)$ is an integer so $(n+1)^2$ is always a square number

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**20** Prove algebraically that the sum of three consecutive even numbers is always a multiple of 6. **[2 marks]**

$(2n) + (2n+2) + (2n+4)$

$= 6n + 6$

$= 6(n+1)$

$(n+1)$ is an integer

so $6(n+1)$ is a multiple of 6

**21** Prove algebraically that the difference between the squares of two consecutive even numbers is always a multiple of 4 **[3 marks]**

$(2n+2)^2 - (2n)^2$

$= 4n^2 + 4n + 4n + 4 - 4n^2$

$= 8n + 4$

$= 4(2n+1)$

$(2n+1)$ is an integer

so $4(2n+1)$ is a multiple of 4

**22** Prove algebraically that the sum of the squares of three consecutive integers is one less than a multiple of 3. **[4 marks]**

$(n)^2 + (n+1)^2 + (n+2)^2$

$= (n^2) + (n^2+2n+1) + (n^2+4n+4)$

$= 3n^2 + 6n + 5$

$= 3n^2 + 6n + 6 - 1$

$= 3(n^2+2n+2) - 1$

$(n^2+2n+2)$ is an integer

so $3(n^2+2n+2)$ is a multiple of 3

and $3(n^2+2n+2) - 1$ is one less than a multiple of 3.

**23** Prove algebraically that the difference between the squares of consecutive integers is equal to the sum of the two integers. **[3 marks]**

Let $n, n+1$ be consecutive integers

$(n+1)^2 - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n + 1$

$= (n) + (n+1)$

$(n)+(n+1)$ is the sum of the two integers

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Turn over ►

**24** Prove algebraically that the product of two consecutive odd numbers is one less than a multiple of 4.

**[3 marks]**

$(2n+1)(2n+3)$

$= 4n^2 + 2n + 6n + 3$

$= 4n^2 + 8n + 3$

$= 4n^2 + 8n + 4 - 1$

$= 4(n^2 + 2n + 1) - 1$

$(n^2+2n+1)$ is an integer

so $4(n^2+2n+1)$ is a multiple of 4

and $4(n^2+2n+1) - 1$ is one less than a multiple of 4

**25** Prove algebraically that the product of three consecutive even numbers is always a multiple of 8.

**[3 marks]**

$2n(2n+2)(2n+4)$

$= 2n(4n^2 + 8n + 4n + 8)$

$= 2n(4n^2 + 12n + 8)$

$= 8n^3 + 24n^2 + 16n$

$= 8(n^3 + 3n^2 + 2n)$

$(n^3+3n^2+2n)$ is an integer

so $8(n^3+3n^2+2n)$ is a multiple of 8

**26** $a$ and $b$ are positive integers.
$a$ is two more than a multiple of 5.
$b$ is two less than a multiple of 5.

Prove that $ab$ is one more than a multiple of 5. **[4 marks]**

Let $a = 5n + 2$ and $b = 5m - 2$

$ab = (5n+2)(5m-2)$

$= 25mn - 10n + 10m - 4$

$= 25mn - 10n + 10m - 5 + 1$

$= 5(5mn - 2n + 2m - 1) + 1$

$(5mn - 2n + 2m - 1)$ is an integer

so $5(5mn - 2n + 2m - 1)$ is a multiple of 5

and $5(5mn - 2n + 2m - 1) + 1$ is one more than a multiple of 5.

**27** Prove that the sum of the squares of three consecutive integers is equal to five more than three times the product of the largest and smallest of the three integers.
**[3 marks]**

Let $n, n+1, n+2$ be consecutive integers

$n^2 + (n+1)^2 + (n+2)^2$

$= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4)$

$= 3n^2 + 6n + 5$

$= 3(n)(n+2) + 5$

$(n)$ is the smallest of the integers and $(n+2)$ is the largest so $3(n)(n+2) + 5$ is 5 more than 3 times their product.

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