Algebraic Proof

SCAN ME

REVISE THIS TOPIC

1 Prove that $(n + 5)^2 - (3n + 8) = (n + 3)(n + 4) + 5$ **[3 marks]**

$(n+5)(n+5) - (3n+8)$

$= n^2 + 5n + 5n + 25 - 3n - 8$

$= n^2 + 7n + 17$

$= n^2 + 7n + 12 + 5$

$= (n+3)(n+4) + 5$

2 Prove that $(2n - 1)^2 - (n - 3)^2 = (3n - 1)(n + 1) - 7$ **[3 marks]**

$(2n-1)(2n-1) - (n-3)(n-3)$

$= (4n^2 - 2n - 2n + 1) - (n^2 - 3n - 3n + 9)$

$= 4n^2 - 4n + 1 - n^2 + 3n + 3n - 9$

$= 3n^2 + 2n - 8$

$= 3n^2 + 2n - 1 - 7$

$= (3n-1)(n+1) - 7$

$\overline{\quad 6}$

3 Prove that $(3n-5)^2 - 2(4n-5)(n-3) = (n+5)(n-1)$ **[3 marks]**

$(3n-5)(3n-5) - 2(4n-5)(n-3)$

$= (9n^2 - 15n - 15n + 25) - 2(4n^2 - 12n - 5n + 15)$

$= 9n^2 - 30n + 25 - 8n^2 + 24n + 10n - 30$

$= n^2 + 4n - 5$

$= (n+5)(n-1)$

4 Prove that $(n-3)^2 - (15+n)(15-n) = 2(n-12)(n+9)$ **[3 marks]**

$(n-3)(n-3) - (15+n)(15-n)$

$= (n^2 - 3n - 3n + 9) - (225 - 15n + 15n - n^2)$

$= n^2 - 6n + 9 - 225 + 15n - 15n + n^2$

$= 2n^2 - 6n - 216$

$= 2(n^2 - 3n - 108)$

$= 2(n-12)(n+9)$

5 n is an integer such that $n > 3$

Prove algebraically that $(n-2)^2 - (n-5)^2$ is always a multiple of 3. **[3 marks]**

$(n-2)(n-2) - (n-5)(n-5)$

$= (n^2 - 2n - 2n + 4) - (n^2 - 5n - 5n + 25)$

$= n^2 - 4n + 4 - n^2 + 5n + 5n - 25$

$= 6n - 21$

$= 3(2n - 7)$

$(2n-7)$ is an integer

so $3(2n-7)$ is a multiple of 3

6 n is a positive integer.

Prove algebraically that $(3n+1)^2 - (3n-4)^2$ is always a multiple of 15. **[3 marks]**

$(3n+1)(3n+1) - (3n-4)(3n-4)$

$= (9n^2 + 3n + 3n + 1) - (9n^2 - 12n - 12n + 16)$

$= 9n^2 + 6n + 1 - 9n^2 + 12n + 12n - 16$

$= 30n - 15$

$= 15(2n - 1)$

$(2n-1)$ is an integer

so $15(2n-1)$ is a multiple of 15

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7 n is a positive integer.
Prove algebraically that $(2n + 5)^2 - (2n + 1)^2$ is always a multiple of 8 **[3 marks]**

$(2n+5)(2n+5) - (2n+1)(2n+1)$

$= (4n^2 + 10n + 10n + 25) - (4n^2 + 2n + 2n + 1)$

$= 4n^2 + 20n + 25 - 4n^2 - 2n - 2n - 1$

$= 16n + 24$

$= 8(2n + 3)$

$(2n+3)$ is an integer
so $8(2n+3)$ is a multiple of 8

8 n is a positive integer such that $n > 2$
Prove algebraically that $(2n + 3)^2 + (3 - n)^2 - (n + 5)^2$ is always one more than a multiple of 4. **[4 marks]**

$(2n+3)(2n+3) + (3-n)(3-n) - (n+5)(n+5)$

$= (4n^2 + 6n + 6n + 9) + (9 - 3n - 3n + n^2) - (n^2 + 5n + 5n + 25)$

$= 4n^2 + 12n + 9 + 9 - 6n + n^2 - n^2 - 10n - 25$

$= 4n^2 - 4n - 7$

$= 4n^2 - 4n - 8 + 1$

$= 4(n^2 - n - 2) + 1$

$(n^2 - n - 2)$ is an integer so $4(n^2-n-2)$ is a multiple of 4. So $4(n^2-n-2)+1$ is one more than a multiple of 4.

9 Prove algebraically that the sum of five consecutive positive integers is always a multiple of 5. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3) + (n+4)$

$= 5n + 10$

$= 5(n+2)$

$(n+2)$ is an integer

so $5(n+2)$ is a multiple of 5

10 Arjan says:

"The sum of four consecutive positive integers is always a multiple of 4".

Use an algebraic method to prove that Arjun is incorrect. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3)$

$= 4n + 6$

$4n+6$ is not a multiple of 4 since 6 is not a multiple of 4.

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11 Prove algebraically that the sum of six consecutive positive integers is always a multiple of 3. **[2 marks]**

$(n) + (n+1) + (n+2) + (n+3) + (n+4) + (n+5)$

$= 6n + 15$

$= 3(2n + 5)$

$2n + 5$ is an integer

so $3(2n+5)$ is a multiple of 3

12 n is a positive integer.

Prove that $(4n - 3)^2 - 3(5n - 3)(n - 1)$ is always a square number. **[3 marks]**

$(4n-3)(4n-3) - 3(5n-3)(n-1)$

$= (16n^2 - 12n - 12n + 9) - 3(5n^2 - 3n - 5n + 3)$

$= 16n^2 - 24n + 9 - 15n^2 + 9n + 15n - 9$

$= n^2$

n^2 is a square number

13 n is a positive integer.
Prove that $(3n + 1)(9n^2 - 3n + 1)$ is always 1 more than a cube number. **[4 marks]**

$(3n+1)(9n^2-3n+1)$
$= 27n^3 - 9n^2 + 3n + 9n^2 - 3n + 1$
$= 27n^3 + 1$
$= (3n)^3 + 1$

$(3n)^3$ is a cube number
so $(3n)^3 + 1$ is one more than a cube number

14 n is a positive integer.
Prove that $(n + 2)^3 - n^3$ is always even. **[4 marks]**

$(n+2)(n+2)(n+2) - n^3$
$= (n^2+4n+4)(n+2) - n^3$
$= n^3 + 4n^2 + 4n + 2n^2 + 8n + 8 - n^3$
$= n^3 + 6n^2 + 12n + 8 - n^3$
$= 6n^2 + 12n + 8$
$= 2(3n^2 + 6n + 4)$
$(3n^2+6n+4)$ is an integer
so $2(3n^2+6n+4)$ is a multiple of 2
and is therefore even

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Turn over ►

15 n is an integer.
Prove that $n^2 - 6n + 10$ is always positive. **[3 marks]**

$$n^2 - 6n + 10$$
$$= (n-3)^2 - 9 + 10$$
$$= (n-3)^2 + 1$$

$(n-3)^2 \geq 0$ and $1 > 0$

so $(n-3)^2 + 1 > 0$ (always positive)

16 n is an integer.
Prove that $n^2 + 3n + 3$ is always positive. **[3 marks]**

$$n^2 + 3n + 3$$
$$= \left(n + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 3$$
$$= \left(n + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{12}{4}$$
$$= \left(n + \tfrac{3}{2}\right)^2 + \tfrac{3}{4}$$

$\left(n + \tfrac{3}{2}\right)^2 \geq 0$ and $\tfrac{3}{4} > 0$

so $\left(n + \tfrac{3}{2}\right)^2 + \tfrac{3}{4} > 0$ (always positive)

17 n is an integer.
Prove that $2n - n^2 - 2$ is always negative. **[3 marks]**

$$2n - n^2 - 2 = -\left[n^2 - 2n + 2\right]$$
$$= -\left[(n-1)^2 - 1 + 2\right]$$
$$= -\left[(n-1)^2 + 1\right]$$
$$= -(n-1)^2 - 1$$

$-(n-1)^2 \leq 0$ and $-1 < 0$

so $-(n-1)^2 - 1 < 0$ (always negative)

18 n and m are consecutive integers and $m > n$.
Prove algebraically that $m^2 - n^2$ is always an odd number. **[3 marks]**

Let $m = n+1$

$m^2 - n^2 = (n+1)^2 - n^2$

$= (n+1)(n+1) - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n + 1$

$2n$ is always even

so $2n+1$ is always odd.

19 n and m are consecutive integers and $m > n$.
Prove algebraically that $mn + m$ is always a square number. **[3 marks]**

Let $m = n+1$

$mn + m = (n+1)n + (n+1)$

$= n^2 + n + n + 1$

$= n^2 + 2n + 1$

$= (n+1)^2$

$(n+1)$ is an integer so $(n+1)^2$ is always a square number

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20 Prove algebraically that the sum of three consecutive even numbers is always a multiple of 6. **[2 marks]**

$(2n) + (2n+2) + (2n+4)$

$= 6n + 6$

$= 6(n+1)$

$(n+1)$ is an integer

so $6(n+1)$ is a multiple of 6

21 Prove algebraically that the difference between the squares of two consecutive even numbers is always a multiple of 4 **[3 marks]**

$(2n+2)^2 - (2n)^2$

$= 4n^2 + 4n + 4n + 4 - 4n^2$

$= 8n + 4$

$= 4(2n+1)$

$(2n+1)$ is an integer

so $4(2n+1)$ is a multiple of 4

22 Prove algebraically that the sum of the squares of three consecutive integers is one less than a multiple of 3. **[4 marks]**

$(n)^2 + (n+1)^2 + (n+2)^2$

$= (n^2) + (n^2 + 2n + 1) + (n^2 + 4n + 4)$

$= 3n^2 + 6n + 5$

$= 3n^2 + 6n + 6 - 1$

$= 3(n^2 + 2n + 1) - 1$

$(n^2 + 2n + 1)$ is an integer

so $3(n^2 + 2n + 1)$ is a multiple of 3

and $3(n^2 + 2n + 1) - 1$ is one less than a multiple of 3.

23 Prove algebraically that the difference between the squares of consecutive integers is equal to the sum of the two integers. **[3 marks]**

Let $n, n+1$ be consecutive integers

$(n+1)^2 - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n + 1$

$= (n) + (n+1)$

$(n) + (n+1)$ is the sum of the two integers

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24 Prove algebraically that the product of two consecutive odd numbers is one less than a multiple of 4.

[3 marks]

$(2n+1)(2n+3)$

$= 4n^2 + 2n + 6n + 3$

$= 4n^2 + 8n + 3$

$= 4n^2 + 8n + 4 - 1$

$= 4(n^2 + 2n + 1) - 1$

(n^2+2n+1) is an integer

so $4(n^2+2n+1)$ is a multiple of 4

and $4(n^2+2n+1) - 1$ is one less than a multiple of 4

25 Prove algebraically that the product of three consecutive even numbers is always a multiple of 8.

[3 marks]

$2n(2n+2)(2n+4)$

$= 2n(4n^2 + 8n + 4n + 8)$

$= 2n(4n^2 + 12n + 8)$

$= 8n^3 + 24n^2 + 16n$

$= 8(n^3 + 3n^2 + 2n)$

(n^3+3n^2+2n) is an integer

so $8(n^3+3n^2+2n)$ is a multiple of 8

26 a and b are positive integers.
a is two more than a multiple of 5.
b is two less than a multiple of 5.

Prove that ab is one more than a multiple of 5. **[4 marks]**

Let $a = 5n + 2$ and $b = 5m - 2$

$ab = (5n+2)(5m-2)$

$= 25mn - 10n + 10m - 4$

$= 25mn - 10n + 10m - 5 + 1$

$= 5(5mn - 2n + 2m - 1) + 1$

$(5mn - 2n + 2m - 1)$ is an integer

so $5(5mn - 2n + 2m - 1)$ is a multiple of 5

and $5(5mn - 2n + 2m - 1) + 1$ is one more than a multiple of 5.

27 Prove that the sum of the squares of three consecutive integers is equal to five more than three times the product of the largest and smallest of the three integers.

[3 marks]

Let n, $n+1$, $n+2$ be consecutive integers

$n^2 + (n+1)^2 + (n+2)^2$

$= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4)$

$= 3n^2 + 6n + 5$

$= 3(n)(n+2) + 5$

(n) is the smallest of the integers and $(n+2)$ is the largest so $3(n)(n+2) + 5$ is 5 more than 3 times their product.

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